

# The difference and ratio of the fractional matching number and the matching number of graphs

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## Abstract

Given a graph  $G$ , the *matching number* of  $G$ , written  $\alpha'(G)$ , is the maximum size of a matching in  $G$ , and the *fractional matching number* of  $G$ , written  $\alpha'_f(G)$ , is the maximum size of a fractional matching of  $G$ . In this paper, we prove that if  $G$  is an  $n$ -vertex connected graph that is neither  $K_1$  nor  $K_3$ , then  $\alpha'_f(G) - \alpha'(G) \leq \frac{n-2}{6}$  and  $\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3n}{2n+2}$ . Both inequalities are sharp, and we characterize the infinite family of graphs where equalities hold.

## 1 Introduction

For undefined terms, see [5]. Throughout this paper,  $n$  will always denote the number of vertices of a given graph. A *matching* in a graph is a set of pairwise disjoint edges. A *perfect matching* in a graph  $G$  is a matching in which each vertex has an incident edge in the matching; its size must be  $n/2$ , where  $n = |V(G)|$ . A *fractional matching* of  $G$  is a function  $\phi : E(G) \rightarrow [0, 1]$  such that for each vertex  $v$ ,  $\sum_{e \in \Gamma(v)} \phi(e) \leq 1$ , where  $\Gamma(v)$  is the set of edges incident to  $v$ , and the *size of a fractional matching*  $\phi$  is  $\sum_{e \in E(G)} \phi(e)$ . Given a graph  $G$ , the *matching number* of  $G$ , written  $\alpha'(G)$ , is the maximum size of a matching in  $G$ , and

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the *fractional matching number* of  $G$ , written  $\alpha'_f(G)$ , is the maximum size of a fractional matching of  $G$ .

Given a fractional matching  $\phi$ , since  $\sum_{e \in \Gamma(v)} \phi(e) \leq 1$  for each vertex  $v$ , we have that  $2 \sum_{e \in E(G)} \phi(e) \leq n$ , which implies  $\alpha'_f(G) \leq n/2$ . By viewing every matching as a fractional matching it follows that  $\alpha'_f(G) \geq \alpha'(G)$  for every graph  $G$ , but equality need not hold. For example, the fractional matching number of a  $k$ -regular graph equals  $n/2$  by setting weight  $1/k$  on each edge, but the matching number of a  $k$ -regular graph can be much smaller than  $n/2$ . Thus it is a natural question to find the largest difference between  $\alpha'_f(G)$  and  $\alpha'(G)$  in a (connected) graph.

In Section 3 and Section 4, we prove tight upper bounds on  $\alpha'_f(G) - \alpha'(G)$  and  $\frac{\alpha'_f(G)}{\alpha'(G)}$ , respectively, for an  $n$ -vertex connected graph  $G$ , and we characterize the infinite family of graphs achieving equality for both results. As corollaries of both results, we have upper bounds on both  $\alpha'_f(G) - \alpha'(G)$  and  $\frac{\alpha'_f(G)}{\alpha'(G)}$  for an  $n$ -vertex graph  $G$ , and we characterize the graphs achieving equality for both bounds.

Our proofs use the famous Berge–Tutte Formula [1] for the matching number as well as its fractional analogue. We also use the fact that there is a fractional matching  $\phi$  for which  $\sum_{e \in E(G)} \phi(e) = \alpha'_f(G)$  such that  $\phi(e) \in \{0, 1/2, 1\}$  for every edge  $e$ , and some refinements of the fact. We can prove both Theorem 6 and Theorem 8 with two different techniques, and for the sake of the readers we demonstrate each method in the proofs of Theorem 6 and Theorem 8.

## 2 Tools

In this section, we introduce the tools we used to prove the main results. To prove Theorem 6, we use Theorem 1 and Theorem 2. For a graph  $H$ , let  $o(H)$  denote the number of components of  $H$  with an odd number of vertices. Given a graph  $G$  and  $S \subseteq V(G)$ , define the *deficiency*  $\text{def}(S)$  by  $\text{def}(S) = o(G - S) - |S|$ , and let  $\text{def}(G) = \max_{S \subseteq V(G)} \text{def}(S)$ . Theorem 1 is the famous Berge–Tutte formula, which is a general version of Tutte’s 1-factor Theorem [4].

**Theorem 1** ([1]). *For any  $n$ -vertex graph  $G$ ,  $\alpha'(G) = \frac{1}{2} (n - \text{def}(G))$ .*

For the fractional analogue of the Berge–Tutte formula, let  $i(H)$  denote the number of isolated vertices in  $H$ . Given a graph  $G$  and  $S \subseteq V(G)$ , let  $\text{def}_f(S) = i(G - S) - |S|$  and  $\text{def}_f(G) = \max_{S \subseteq V(G)} \text{def}_f(S)$ . Theorem 2 is the fractional version of the Berge–Tutte Formula. This is also the fractional analogue of Tutte’s 1-Factor Theorem saying that  $G$  has a fractional perfect matching if and only if  $i(G - S) \leq |S|$  for all  $S \subseteq V(G)$  (implicit in Pulleyblank [2]), where a fractional perfect matching is a fractional matching  $f$  such that  $2 \sum_{e \in E(G)} f(e) = n$ .

**Theorem 2** ([3] See Theorem 2.2.6). *For any  $n$ -vertex graph  $G$ ,  $\alpha'_f(G) = \frac{1}{2}(n - \text{def}_f(G))$ .*

When we characterize the equalities in the bounds of Theorem 6 and Theorem 8, we need the following proposition. Recall that  $G[S]$  is the graph induced by a subset of the vertex set  $S$ .

**Proposition 3** ([3] See Proposition 2.2.2). *The following are equivalent for a graph  $G$ .*

- (a)  *$G$  has a fractional perfect matching.*
- (b) *There is a partition  $\{V_1, \dots, V_n\}$  of the vertex set  $V(G)$  such that, for each  $i$ , the graph  $G[V_i]$  is either  $K_2$  or Hamiltonian.*
- (c) *There is a partition  $\{V_1, \dots, V_n\}$  of the vertex set  $V(G)$  such that, for each  $i$ , the graph  $G[V_i]$  is either  $K_2$  or Hamiltonian graph on an odd number of vertices.*

Theorem 4 and Observation 5 are used to prove Theorem 8.

**Theorem 4** ([3] See Theorem 2.1.5). *For any graph  $G$ , there is a fractional matching  $f$  for which*

$$\sum_{e \in E(G)} f(e) = \alpha'_f(G)$$

*such that  $f(e) \in \{0, 1/2, 1\}$  for every edge  $e$ .*

Given a fractional matching  $f$ , an *unweighted* vertex  $v$  is a vertex with  $\sum_{e \in \Gamma(v)} f(e) = 0$ , and a *full* vertex  $v$  is a vertex with  $f(vw) = 1$  for some vertex  $w$ . Note that  $w$  is also a full vertex. An  *$i$ -edge*  $e$  is an edge with  $f(e) = i$ . Note that the existence of an 1-edge guarantees the existence of two full vertices. A vertex subset  $S$  of a graph  $G$  is *independent* if  $E(G[S]) = \emptyset$ , where  $G[S]$  is the graph induced by  $S$ .

**Observation 5.** *Among all the fractional matchings of an  $n$ -vertex graph  $G$  satisfying the conditions of Theorem 4, let  $f$  be a fractional matching with the greatest number of edges  $e$  with  $f(e) = 1$ . Then we have the following:*

- (a) *The graph induced by the  $\frac{1}{2}$ -edges is the union of odd cycles. Furthermore, if  $C$  and  $C'$  are two disjoint cycles in the graph induced by  $\frac{1}{2}$ -edges, then there is no edge  $uu'$  such that  $u \in V(C)$  and  $u' \in V(C')$ .*
- (b) *The set  $S$  of the unweighted vertices is independent. Furthermore, every unweighted vertex is adjacent only to a full vertex.*
- (c)  *$\alpha'_f(G) \geq w_1 + \sum_{i=1}^{\infty} ic_i$ ,  $\alpha'_f(G) = w_1 + \sum_{i=1}^{\infty} (\frac{2i+1}{2})c_i$ , and  $n = w_0 + 2w_1 + \sum_{i=1}^{\infty} (2i+1)c_i$ , where  $w_0$ ,  $w_1$ , and  $c_i$  are the number of unweighted vertices, the number of 1-edges, and the number of odd cycles of length  $2i+1$  in the graph induced by  $\frac{1}{2}$ -edges in  $G$ , respectively.*

*Proof.* (a) The graph induced by the  $\frac{1}{2}$ -edges cannot have a vertex with degree at least 3 since  $\sum_{e \in \Gamma(v)} f(e) \leq 1$  for each vertex  $v$ . Thus the graph must be a disjoint union of paths

or cycles. If the graph contains a path or an even cycle, then by replacing weight  $1/2$  on each edge on the path or the even cycle with weight 1 and 0 alternatively, we can have a fractional matching with the same fractional matching number and more edges with weight 1, which contradicts the choice of  $f$ . Thus the graph induced by the  $\frac{1}{2}$ -edges is the union of odd cycles. If there is an edge  $uv$  such that  $u \in V(C)$  and  $v \in V(C')$ , where  $C$  and  $C'$  are two different odd cycles induced by some  $\frac{1}{2}$ -edges, then  $f(uv) = 0$ , since  $\sum_{e \in \Gamma(x)} f(e) \leq 1$  for each vertex  $x$ . By replacing weights 0 and  $1/2$  on the edge  $uv$  and the edges on  $C$  and  $C'$  with weight 1 on  $uv$ , and 0 and 1 on the edges in  $E(C)$  and  $E(C')$  alternatively, not violating the definition of a fractional matching, we have a fractional matching with the same fractional matching number with more edges with weight 1, which is a contradiction. Thus we have the desired result.

(b) If two unweighted vertices  $u$  and  $v$  are adjacent, then we can put a positive weight on the edge  $uv$ , which contradicts the choice of  $f$ . If there exists an unweighted vertex  $x$ , which is not incident to any full vertex, then  $x$  must be adjacent to a vertex  $y$  such that  $f(yy_1) = 1/2$  and  $f(yy_2) = 1/2$  for some vertices  $y_1$  and  $y_2$ . By replacing the weights 0,  $1/2$ , and  $1/2$  on  $xy$ ,  $yy_1$ , and  $yy_2$  with 1, 0, 0, respectively, we have a fractional matching with the same fractional matching number with more edges with weight 1, which is a contradiction.

(c) By the definitions of  $w_0$ ,  $w_1$ , and  $c_i$ , we have the desired result.  $\square$

### 3 Sharp upper bound for $\alpha'_f(G) - \alpha'(G)$

What are the structures of the graphs having the maximum difference between the fractional matching number and the matching number in an  $n$ -vertex connected graph? The graphs may have big fractional matching number and small matching number. So, by the Berge-Tutte Formula and its fractional version, they may have a vertex subset  $S$  such that almost all of the odd components of  $G - S$  have at least three vertices in order to get  $S$  to have small fractional deficiency and big deficiency. This is our idea behind the proof of Theorem 6.

**Theorem 6.** *For  $n \geq 5$ , if  $G$  is a connected graph with  $n$  vertices, then  $\alpha'_f(G) - \alpha'(G) \leq \frac{n-2}{6}$ , and equality holds only when either*

- (i)  $n = 5$  and either  $C_5$  is subgraph of  $G$  or  $K_2 + K_3$  is a subgraph of  $G$ , or
- (ii)  $G$  has a vertex  $v$  such that the components of  $G - v$  are all  $K_3$  except one single vertex.

*Proof.* Among all the vertex subsets with maximum deficiency, let  $S$  be the largest set. By the Berge-Tutte Formula,  $\alpha'(G) = \frac{1}{2}(n - \text{def}(S))$ , and by the choice of  $S$ , all components of  $G - S$  have an odd number of vertices. Let  $x$  be the number of isolated vertices of  $G - S$ , and let  $y$  be the number of other components of  $G - S$ . This implies  $n \geq |S| + x + 3y$ . If  $S = \emptyset$ , then  $\alpha'(G) \in \{\frac{n}{2}, \frac{n-1}{2}\}$ , depending on the parity of  $n$ . In this case,  $\alpha'_f(G) - \alpha'(G) \leq$

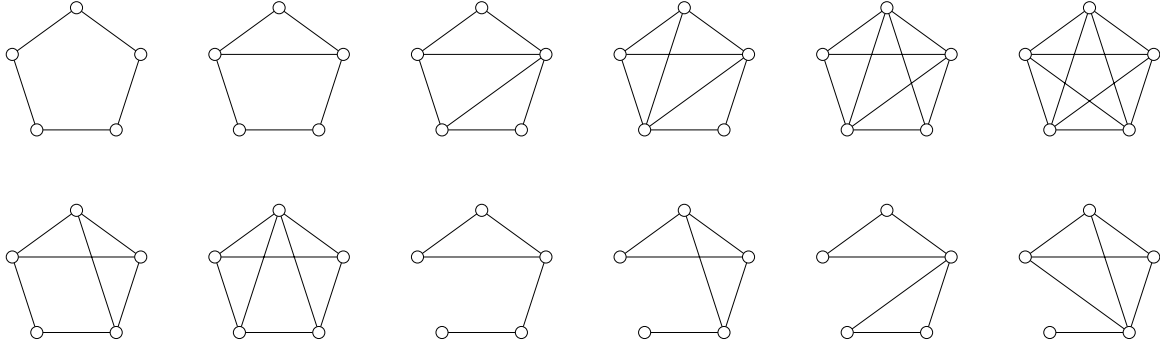


Figure 1: All 5-vertex graphs in Theorem 6 (i) and Theorem 8 (i)

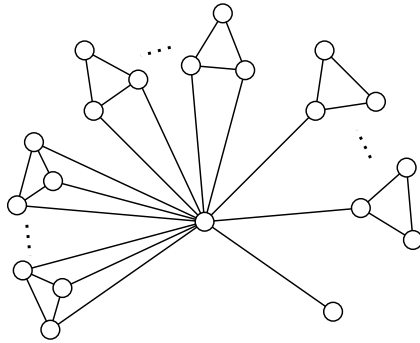


Figure 2: All graphs in Theorem 6 (ii) and Theorem 8 (ii)

$\frac{n}{2} - \frac{n-1}{2} = \frac{1}{2} \leq \frac{n-2}{6}$ , since  $n \geq 5$ . Now, assume that  $S$  is non-empty.

*Case 1:*  $x = 0$ . Since  $\text{def}_f(G) \geq 0$ ,  $|S| \geq 1$ , and  $n \geq |S| + 3y$ , we have

$$\begin{aligned} \alpha'_f(G) - \alpha'(G) &= \frac{1}{2}(n - \text{def}_f(G)) - \frac{1}{2}(n - \text{def}(S)) = \frac{1}{2}(\text{def}(S) - \text{def}_f(G)) \\ &\leq \frac{1}{2}(y - |S| - 0) \leq \frac{1}{2}\left(\frac{n - |S|}{3} - |S|\right) = \frac{n - 4|S|}{6} \leq \frac{n - 4}{6} < \frac{n - 2}{6}. \end{aligned}$$

*Case 2:*  $x \geq 1$ . Since  $n \geq |S| + x + 3y$ ,  $|S| \geq 1$ , and  $x \geq 1$ , we have

$$\begin{aligned} \alpha'_f(G) - \alpha'(G) &= \frac{1}{2}(n - \text{def}_f(G)) - \frac{1}{2}(n - \text{def}(S)) = \frac{1}{2}(\text{def}(S) - \text{def}_f(G)) \\ &\leq \frac{1}{2}(x + y - |S| - (x - |S|)) \leq \frac{y}{2} = \frac{n - x - |S|}{6} \leq \frac{n - 2}{6}. \end{aligned}$$

Equality in the bound requires equality in each step of the computation. When  $n = 5$ , we conclude that (i) follows by Proposition 3. In Case 1, we cannot have equality, and in Case 2, we have  $|S| = 1$ ,  $x = 1$ , and  $n = |S| + x + 3y = 2 + 3y$ . Since  $G$  is connected, the components of  $G - S$  are  $P_3$  or  $K_3$  except only one single vertex. If a component of  $G - S$  is a copy of  $P_3$ , then by choosing the central vertex  $u$  of the path, we have  $\text{def}(S \cup \{u\}) = o(G - (S \cup \{u\})) - |S \cup \{u\}| = o(G - S) - |S|$ , yet  $|S \cup \{u\}| > |S|$ , which contradict the choice of  $S$ . Thus we have the desired result.  $\square$

**Corollary 7.** *For any  $n$ -vertex graph  $G$ , we have  $\alpha'_f(G) - \alpha'(G) \leq \frac{n}{6}$ , and equality holds only when  $G$  is the disjoint union of copies of  $K_3$ .*

*Proof.* First, we show that if  $n \leq 4$  and  $G$  is connected, then  $\alpha'_f(G) - \alpha'(G) \leq \frac{n}{6}$ , and equality holds only when  $G = K_3$ . If  $n \leq 2$ , then  $G \in \{K_1, K_2\}$ , which implies that  $\alpha'_f(G) - \alpha'(G) = 0 < n/6$ . If  $n = 3$ , then  $G \in \{P_3, K_3\}$ . Note that  $\alpha'_f(P_3) - \alpha'(P_3) = 1 - 1 = 0 < 3/6$  and  $\alpha'_f(K_3) - \alpha'(K_3) = 3/2 - 1 = 1/2 \leq 3/6$ . Furthermore, equality holds only when  $G = K_3$ . If  $n = 4$ , then either  $G = K_{1,3}$  or  $G$  contains  $P_4$  as a subgraph. Since  $\alpha'_f(K_{1,3}) - \alpha'(K_{1,3}) = 1 - 1 = 0 < 4/6$  and  $\alpha'_f(P_4) - \alpha'(P_4) = 2 - 2 = 0 < 4/6$ , we conclude that for any positive integer  $n$ ,  $\alpha'_f(G) - \alpha'(G) \leq \frac{n}{6}$ . In fact, if  $n \geq 5$ , then by Theorem 6, the difference must be at most  $\frac{n-2}{6}$ . Thus, for connected graphs, equality holds only when  $G = K_3$ .

Now, if we assume that  $G$  is disconnected, then  $G$  is the disjoint union of connected graphs  $G_1, \dots, G_k$ . Let  $|V(G_i)| = n_i$  for  $i \in [k]$ . Since

$$\begin{aligned} \alpha'_f(G) - \alpha'(G) &= [\alpha'_f(G_1) + \dots + \alpha'_f(G_k)] - [\alpha'(G_1) + \dots + \alpha'(G_k)] \\ &= [\alpha'_f(G_1) - \alpha'(G_1)] + \dots + [\alpha'_f(G_k) - \alpha'(G_k)] \leq \frac{n_1}{6} + \dots + \frac{n_k}{6} = \frac{n}{6}, \end{aligned}$$

equality holds only when each  $G_i$  is a copy of  $K_3$  for  $i \in [k]$ .  $\square$

## 4 Sharp upper bound for $\frac{\alpha'_f(G)}{\alpha'(G)}$

To prove the upper bound of Theorem 8, we still can use the Berge-Tutte formula and its fractional analogue. However, we provide an alternative way to prove the theorem.

**Theorem 8.** *For  $n \geq 5$ , if  $G$  is a connected graph with  $n$  vertices, then  $\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3n}{2n+2}$ , and equality holds only when either*

- (i)  $n = 5$  and either  $C_5$  is a subgraph of  $G$  or  $K_2 + K_3$  is a subgraph of  $G$ , or
- (ii)  $G$  has a vertex  $v$  such that the components of  $G - v$  are all  $K_3$  except one single vertex.

*Proof.* Among all the fractional matchings of an  $n$ -vertex graph  $G$  with the size equal to  $\alpha'_f(G)$ , let  $f$  be a fractional matching such that the number of edges  $e$  with  $f(e) = 1$  is maximized. We follow the notation in Observation 5.

*Case 1:*  $w_0 = w_1 = 0$ . Since  $G$  is connected and  $n \geq 5$ , there exists only one  $i$  such that  $i \geq 2$  and  $c_i$  is not zero, and  $\alpha'(G) = ic_i \neq 0$ . Then we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{\left(\frac{2i+1}{2}\right)c_i}{ic_i} = 1 + \frac{1}{2i} \leq \frac{5}{4}.$$

*Case 2:*  $w_0 \geq 1$  and  $w_1 = 0$ . By part (b) of Observation 5, this cannot happen.

*Case 3:*  $w_0 = 0$  and  $w_1 \geq 1$ . Since  $\sum_{i=1}^{\infty} c_i \leq \frac{n-2w_1}{3}$ , by part (c) of Observation 5, we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{w_1 + \sum_{i=1}^{\infty} \left(\frac{2i+1}{2}\right)c_i}{w_1 + \sum_{i=1}^{\infty} ic_i} = \frac{\frac{n-w_0}{2}}{\frac{n-w_0-\sum_{i=1}^{\infty} c_i}{2}} = \frac{n}{n - \sum_{i=1}^{\infty} c_i} \leq \frac{n}{n - \frac{n-2w_1}{3}} = \frac{3n}{2n + 2w_1} \leq \frac{3n}{2n + 2}.$$

*Case 4:*  $w_0 \geq 1$  and  $w_1 \geq 1$ . Since  $\sum_{i=1}^{\infty} c_i \leq \frac{n-2w_1-w_0}{3}$ , by part (c) of Observation 5, we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{\frac{n-w_0}{2}}{\frac{n-w_0-\sum_{i=1}^{\infty} c_i}{2}} \leq \frac{n-w_0}{n-w_0-\frac{n-2w_1-w_0}{3}} = \frac{3(n-w_0)}{2(n+w_1-w_0)} < \frac{3n}{2(n+w_1)} \leq \frac{3n}{2(n+1)}.$$

Equality in the bound requires equality in each step of the computation; we only need to check Case 1 and Case 2. In Case 1, we have  $i = 2$ , which means that  $n = 5$  and  $G$  contains a copy of  $C_5$ . In Case 3, we have  $w_1 = 1$  and  $\sum_{i=1}^{\infty} c_i = \frac{n-2}{3}$ , which means that the graph induced by the  $\frac{1}{2}$ -edges is the union of  $K_3$ . Thus  $G$  has  $K_2 + kK_3$  as a subgraph for some positive integer  $k$ . Note that there is an edge between the copy of  $K_2$  and any copy of  $K_3$  by part (b) of Observation 5. Also, there are no edges between any pair of two triangles by part (a) of Observation 5. Let  $u$  and  $v$  be the two vertices corresponding to the copy of  $K_2$ . If there are two different triangles  $C$  and  $C'$  in  $G$  such that  $u$  and  $v$  are incident to  $C$  and  $C'$ , respectively, then we have  $\alpha'(G) > w_1 + c_1$ , which implies that we cannot have equality in the first inequality in Case 3. Thus, we conclude that  $G$  contains a copy of either  $K_2 + K_3$  as a subgraph or a vertex  $v$  such that the components of  $G - v$  are all  $K_3$  except only one single vertex.  $\square$

**Corollary 9.** *For any  $n$ -vertex graph  $G$  with at least one edge, we have  $\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3}{2}$ , and equality holds only when  $G$  is the disjoint union of copies of  $K_3$ .*

*Proof.* By the proof of Corollary 7, if  $n \leq 4$  and  $G$  is connected, then  $\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3}{2}$ , and equality holds only when  $G = K_3$ . If we assume that  $G$  is disconnected, then  $G$  is the disjoint union of connected graphs  $G_1, \dots, G_k$ . Let  $|V(G_i)| = n_i$  for  $i \in [k]$ . Without loss of generality, we may assume that  $\frac{\alpha'_f(G_1)}{\alpha'(G_1)} \geq \frac{\alpha'_f(G_i)}{\alpha'(G_i)}$  for all  $i \in [k]$ . Then we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} = \frac{\alpha'_f(G_1) + \dots + \alpha'_f(G_k)}{\alpha'(G_1) + \dots + \alpha'(G_k)} \leq \frac{\alpha'_f(G_1)}{\alpha'(G_1)} \leq \frac{3}{2},$$

and equality holds only when each  $G_i$  is a copy of  $K_3$  for  $i \in [k]$ . □

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